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Decomposable Maps in General Tessellation Structures*

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For an arbitrary template T in an m-state d-dimensional tessellation structure, it is shown that there exists local maps on T which cannot be composed from a minmum number of applications of local maps on a simpler template. Further, it is shown that structures with local maps which have this property are in the minority. In particular, for a fixed template T, the fraction of local maps with minimal decompositions approaches 0 as m, the number of states, increases arbitrarily.

1. Introduction

The tessellation structure or cellular space consists of a set of infinitely many finite-state machines interconnected in some regular fashion. A computation takes the structure from some initial configuration to a final configuration and is characterized by speed and complexity. Speed is measured by the number of configurations which occur between the initial and final configurations. Complexity is measured by the number of states in each machine and by the number of interconnections to other machines.

Yamada and Amoroso [1, 2] and Smith [3] have shown that tradeoffs can be made between speed and complexity. That is, given some tessellation structure, it is possible to find another which performs *essentially* the same computation but is perhaps faster or less complex. A tradeoff occurs because a tessellation structure which is faster is generally more complex, for example.

An example of a tradeoff which results in a slower but less complex system occurs in a structure whose local map can be composed of k applications of local maps on a smaller neighborhood. In the latter structure, computations are slower by a factor of k, but the neighborhood is significantly smaller. Amoroso and Epstein [4] have shown that in the set of one-dimensional 2-state tessellation structures, there exists for each neighborhood of $n \geqslant 2$ neighbors, a structure whose local map cannot be composed from a local map on some simpler neighborhood. A counting argument is used here to show that, in the set of d-dimensional m-state structures, there is a structure whose local map cannot have a decomposition called a minimal decomposition on *any* smaller neighborhood map. The results further indicate that for general tessellation structures, maps with minimum decompositions are greatly outnumbered by other maps.

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2. Notation

The notation used is similar to that of Yamada and Amoroso [1, 2]. Let A be a finite set of states and Z^d the set of d-tuples of integers. A finite-state machine G exists at each point in Z^d, and an assignment of states to each machine constitutes a configuration c; that is $c\colon Z^d \to A$. A machine and a point will be called a *cell.* The cell to which some particular cell connects is specified by the neighborhood stencil $X = (i_1, i_2, \ldots, i_n)$ where $i_j \in Z^d$. Line 1 of G at i connects to the machine at $i_1 + i$, line 2 to the machine at $i_2 + i$, etc. Where the order of elements of X is unimportant, it is convenient to use the template T of X (a term introduced by Smith [3]). Thus, if $X = (i_1, i_2, \ldots, i_n)$, then T is the unordered set $T = \{i_1, i_2, \ldots, i_n\}$. The machine at $i_j + i$, $i_j \in T$ is said to be a *neighbor* of the machine at i.

Let C be the set of all configurations and let $\sigma\colon A^n \to A$, the *local map*, denote the next state of a machine at i whose neighbors are $i_j + i$, $i_j \in T$ for all $i \in Z^d$ and $1 \leqslant j \leqslant n$. σ gives rise to a *parallel map* $\tau_{X,\sigma}$ which maps C into itself. That is, $\tau_{X,\sigma}\colon C \to C$, where $c' = \tau_{X,\sigma}(c)$ is the mapping

$$c'(i) = \sigma(c(i_1 + i), c(i_2 + i), \ldots, c(i_n + i)).$$

A special state $q_0 \in A$ has the property $q_0 = \sigma(q_0, q_0, \ldots, q_0)$ for all local maps σ in the cellular system. q_0 is called the *quiescent* state. Its existence guarantees that if there are finitely many nonquiescent cells in c, there will be finitely many nonquiescent states in c'.

For stencils X and X', X' is said to be *simpler* than X, written $X' < X$, if each component of X' is a component of X, and there is some component of X which is not a component of X'.

A parallel map $\tau_{X,\sigma}$ in which σ depends on all components of X is *decomposable* if there exists an X' such that $X' < X$ and local maps $\sigma_1, \sigma_2, \ldots, \sigma_k$ such that

$$\tau_{X,\sigma} = \tau_{X',\sigma_k} \circ \tau_{X',\sigma_{k-1}} \circ \cdots \circ \tau_{X',\sigma_1}, \tag{1}$$

where $|X'| > 1$ and $k > 1$. Otherwise, $\tau_{X,\sigma}$ is said to be *indecomposable.* (1) says that a single application of local map σ over X is equivalent to the application over X' of a local map σ_1, followed by σ_2, etc. If $\tau_{X,\sigma}$ is an (in)decomposable parallel map, then σ is termed an (in)decomposable local map.

A procedure for deciding whether a local map σ is decomposable is to enumerate (1) a neighborhood stencil, and (2) a sequence of local maps $\sigma_1, \sigma_2, \ldots$, and σ_k and to check if the resulting composite map is σ. If it is not, the next stencil-local map sequence is generated and tested, etc. Since there is an effective enumeration of stencil-local map sequences, if σ is decomposable, the procedure will halt successfully. However, if indecomposable, it will continue indefinitely.

As an example, consider a one-dimensional binary cellular system with the three-cell neighborhood index $X = (0, 1, 2)$. Let σ be σ_M, the majority function defined as follows,

$$c'(i) = \sigma_M(c(i), c(i+1), c(i+2)) = c(i) \cdot c(i+1) + c(i) \cdot c(i+2) + c(i+1) \cdot c(i+2)$$

where $\cdot$ and $+$ are the AND and OR operations respectively. That is, the next state of a cell $c'(i)$ is 1 if the present state of two or three of its neighbor is 1. The only X' such that $X' < X_M$ is $X' = (0, 1)$. Enumerating all pairs of local maps produces no decomposable maps identical to σ_M. Thus, τ_{X_M,σ_M} has no decomposition of the form (1) for $k = 2$. Following the procedure outlined above, all three-cell local maps should be examined, etc. Note that for local map sequences of length k the "effective" neighborhood index is $(0, 1, 2,..., k)$. This represents the neighboring cells which a composite map could possibly depend on. If σ_M has a composition for $k \geqslant 3$, however, it must, of course, be independent of 3, 4,..., and k.

Because σ depends on all components of X, there is a minimum value k below which no decomposition can exist because the effective neighborhood produced by k applications of local maps does not include all elements in X. Denote such a k as k_M and denote a decomposition in which $k = k_M$ as *minimal*. For example, σ_M has no minimal decomposition.[1]

Note that if $\tau_{X,\sigma}$ has a (minimal) decomposition of the form (1), it also has the decomposition,

$$\tau_{X,\sigma} = \tau_{X''',\sigma'''} \circ \tau_{X'',\sigma''}$$

where,

$$\tau_{X'',\sigma'''} = \tau_{X',\sigma_{k_M}} \circ \tau_{X',\sigma_{k_{M-1}}} \circ \cdots \circ \tau_{X',\sigma_i},$$

and

$$\tau_{X'',\sigma''} = \tau_{X',\sigma_{i-1}} \circ \tau_{X',\sigma_{i-2}} \circ \cdots \circ \tau_{X',\sigma_1}.$$

But this implies $\tau_{X,\sigma}$ has the decomposition

$$\tau_{X,\sigma} = \tau_{X^*,\sigma_2^*} \circ \tau_{X^*,\sigma_1^*},$$

where $X^* > X''$ or $X^* = X''$ and $X^* > X'''$ or $X^* = X'''$ and where $\sigma_1^*(\sigma_2^*)$ is exactly $\sigma''(\sigma''')$ except that it applies to perhaps a larger neighborhood index $X''(X''')$.[2] Note that this decomposition is both distinct and minimal. Thus if $\tau_{X,\sigma}$ has a minimal decomposition on any X' such that $X' < X$, it also has a distinct minimal composition with exactly two components. This observation will be useful in the next section.

3. Decomposable Maps

Consider an arbitrary d-dimensional tessellation structure with template T. The number $L(m, n)$ of distinct local maps on T is

$$L(m, n) = m^{m^n}, \tag{2}$$

[1] It has been shown (Butler [6]) that in a one-dimensional binary cellular automata system a local map σ has no minimal decomposition on $X' = (0, 1)$ if and only if it is indecomposable. Thus, σ_M, for example, is not decomposable. It is an open question whether this is true of more general cellular systems.

[2] It is assumed that both σ_1^* and σ_2^* operate on the same neighborhood stencil, since an operational tessellation structure is likely to have a fixed interconnection pattern among neighbors (and a local map which can be programmed).

where $m = |A|$, the cardinality of the state set and $n = |T|$, the number of neighbors in T. (2) follows from the fact that there are m^n different ways to assign m states to n neighbors, and for each such assignment, there are m ways to choose the next state. It is of interest to calculate the number of parallel maps.

LEMMA 1. *For each local map σ, there exists a distinct parallel map $\tau_{X,\sigma}$.*

Proof. Suppose, on the contrary, that two local maps σ_1 and σ_2 correspond to the same parallel map $\tau_{X,\sigma_1} = \tau_{X,\sigma_2}$. Since σ_1 and σ_2 are distinct, there is an assignment α of states to a cell and its neighbors such that $\sigma_1(\alpha) \neq \sigma_2(\alpha)$. Consider a configuration c containing α. The new configuration $\tau_{X,\sigma_1}(c)$ and $\tau_{X,\sigma_2}(c)$ must differ at least at one cell, contradicting the assumption that τ_{X,σ_1} and τ_{X,σ_2} are equal. Q.E.D.

It follows from Lemma 1 that the number of parallel maps $P(m, n)$ is identical to $L(m, n)$. Thus,

$$P(m, n) = m^{m^n}. \tag{3}$$

Included in the count of (2) and (3) are trivial maps. For example, one local map included in the count of (2) produces some fixed next state regardless of the present states of a cell and its neighbor.

LEMMA 2. *The number of local maps $L'(m, n)$ in a tessellation structure which depends on all elements of the template T is*

$$L'(m, n) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} m^{m^i}, \tag{4}$$

where $m = |A|$, the cardinality of the state set, and $n = |T|$.

Proof. The proof follows directly from the principle of inclusion/exclusion and is a generalization of a result by Harrison [p. 169, 5]. Q.E.D.

Of the $L'(m, n)$ local maps dependent on all components of the stencil, we are interested in counting the number which have minimal decompositions on any smaller stencil X'. As stated earlier, any local map σ which has a minimal decomposition also has the distinct minimal decomposition

$$\tau_{X,\sigma} = \tau_{X',\sigma_2} \circ \tau_{X',\sigma_1}.$$

Thus, the number of local maps $N_d(m, n)$ which have at least one minimal decomposition can be no greater than the number of compositions of the form shown above. Since the number of components in X' does not exceed $n - 1$, where n is the number of components in X, $N_d(m, n)$ is bounded above as follows:

$$N_d(m, n) < (m^{m^{n-1}})(m^{m^{n-1}}). \tag{5}$$

This is a strict upper bound, because one of the $m^{m^{n-1}}$ choices for σ_2 is trivial, mapping all cells to the same fixed state. For this case, σ is the same regardless of the choice for σ_1.

The number of local maps which have no minimal decomposition $N_{nd}(m, n)$ is,

$$N_{nd}(m, n) = L'(m, n) - N_d(m, n). \tag{6}$$

From (4), (5), and (6) we get a lower bound on $N_{nd}(m, n)$ as follows:

$$N_{nd}(m, n) > \sum_{i=0}^{n} (-1)^{n-1} \binom{n}{i} m^{m^i} - m^{2m^{n-1}}. \tag{7}$$

Since the summation of (4) comes from the principle of inclusion/exclusion, the contribution from all but the two higher order terms is positive. Thus, these terms can be neglected in (7) without affecting the inequality.

$$N_{nd}(m, n) > m^{m^n} - nm^{m^{n-1}} - m^{2m^{n-1}}. \tag{8}$$

Further,

$$m^{m^{n-1}} > n,$$

for $m, n \geqslant 2$, and so (8) can be rewritten as

$$N_{nd}(m, n) > m^{m^n} - 2m^{2m^{n-1}} = (m^{m^{n(1-2/m)}} - 2)\, m^{2m^{n-1}}. \tag{9}$$

For $m, n \geqslant 2$,

$$m^{2m^{n-1}} > 1,$$

and for $m > 2$, $n \geqslant 2$,

$$m^{m^{n(1-2/m)}} > 2.$$

Thus, for $m > 2$, $n \geqslant 2$ (9) shows $N_{nd}(m, n) > 1$. The case for binary tessellation structures can be handled as follows. For $m = 2$,

$$L'(m, n) > 2^{2^n} - 2^{2^{n-1}}.$$

(5), however, gives an upper bound which is too loose since (8) gives a negative value for $m = 2$. For $m = 2$, a better bound is

$$N_d(m, n) < (2^{2^{n-1}} - 2)^2 = 2^{2^n} - 4 \cdot 2^{2^{n-1}} + 4.$$

Here we eliminate, as a possible composite local map, a trivial one in which the next state is 0 or 1 regardless of the previous state. Since $L(2, n) > N_d$, for $n \geqslant 2$, $m = 2$, $N_{nd}(2, n) > 1$. This proves the following:

THEOREM 1. *For each template $T \subset Z^d$ there exists a tessellation structure with a local map on T which has no minimal decomposition.*

An exact expression for $N_{nd}(m, n)$ is difficult to calculate because of the problem of characterizing local maps with decompositions. However, most of what is known indicates that $N_d(m, n)$ is small. In particular, since

$$m^{m^n} - nm^{m^{n-1}} - m^{2m^{n-1}}$$

is a lower bound on $N_{nd}(m, n)$ and m^{m^n} is an upper bound on $L'(m, n)$, the fraction of local maps which are indecomposable is bounded by

$$\frac{N_{nd}(m, n)}{L'(m, n)} \geqslant \frac{m^{m^n} - nm^{m^{n-1}} - m^{2m^{n-1}}}{m^{m^n}} = 1 - nm^{m^{n(1/m-1)}} - m^{m^{n(2/m-1)}}. \tag{10}$$

If m becomes arbitrarily large while n remains fixed, the right side of (10) approaches 1. Thus,

$$\lim_{\substack{m\to\infty \\ n\geqslant 2}} \frac{N_{nd}(m, n)}{L'(m, n)} = 1,$$

and so, for a fixed number of neighbors n, the fraction of local maps which have a minimal decomposition approaches 0 as the number of states increases arbitrarily.

Another indication of the sparsity of decomposable local maps can be seen in the results of a computer program to enumerate decomposable local maps in one-dimensional tessellation structures. When $n = 3$, only 62 local maps have a minimum decomposition. This number includes *all* maps, not just those which satisfy the quiescent condition $q_0 = \sigma(q_0, q_0, \ldots, q_0)$. Of these, 28 maps are independent of at least one of the three neighbors. Thus, only 34 of the 218 maps dependent on all neighbors or 15.6 % have minimum decompositions. Considering local maps which satisfy the quiescent condition that are composed of local maps which also satisfy the quiescent condition, only 19 have a minimum decomposition!

4. Concluding Remarks

The main result presented here is a proof that for any neighborhood template T, there exists a d-dimensional m-state tessellation structure in which the computation performed by the single application of a local map cannot be performed by a minimum number of local maps on a simpler template. It should be noted that this does not contradict Theorem 3.4 of Smith [3] which states that the computation performed by a tessellation structure with an arbitrary template can be *simulated* by a tessellation structure whose template consists of a central cell plus a representative cell from each dimension. The two tessellations in the latter case have generally different configuration sets, while the tessellations described here are restricted to have the same tessellation set.

Finally, it has been shown that maps with minimum decompositions represent a small fraction of the total number of local maps.

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